

Quantum and classical structures in nondeterministic computation

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Abstract. In categorical quantum mechanics, classical structures characterize the classical interfaces of quantum resources on one hand, while on the other hand giving rise to some quantum phenomena. In the standard Hilbert space model of quantum theories, classical structures over a space correspond to its orthonormal bases. In the present paper, we show that classical structures in the category of relations correspond to direct sums of abelian groups. Although relations are, of course, not an interesting model of quantum computation, this result has some interesting computational interpretations. If relations are viewed as denotations of *nondeterministic* programs, it uncovers a wide variety of non-standard quantum structures in this familiar area of classical computation. Ironically, it also opens up a version of what in philosophy of quantum mechanics would be called an ontic-epistemic gap, as it provides no interface to these nonstandard quantum structures.

1 Introduction

Classical structures came to be a useful algebraic tool for analyzing the conceptual foundations of quantum computation [10,7,12]. They characterize the classical interfaces of quantum resources on one hand, and generate entanglement structures, and other essentially quantum phenomena on the other. In the standard, Hilbert space model of quantum theories, classical structures over a space exactly correspond to its orthonormal bases. In nonstandard models, however, they provide a generic conduit to the classical and the quantum features.

Categorical quantum mechanics, initiated in [2], axiomatizes some basic quantum phenomena in the framework of dagger-compact categories. This remarkably rich yet succinct structure has arisen in part from the experience gathered in semantics of programming languages. The most direct source are probably Abramsky's *interaction categories* [3,25], developed to capture the idea of concurrent programs as *relations extended in time*. As a consequence, categories of relations, in all their various flavors arising from various resources [4,5,13,22,23], provide models of categorical quantum mechanics, albeit degenerate because of the trivial dagger structure. Nevertheless, the notion of a classical structure over

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relations is well defined. In the present paper, we provide a complete characterization of classical structures over relations.¹

But what is the relevance and meaning of such a result? Although some relationally based "toy models" of certain quantum phenomena [27] have awakened a lot of interest, a category of relations itself is a rather degenerate model of quantum computation. Its duality and scalar structures in particular seem too simple to accommodate the complex interactions between the quantum and the classical phenomena. — It is therefore only more surprising that, even in this simple framework, classical structures seem to have an interesting story to tell.

Outline of the paper

In section 2, we summarize the definitions of classical and quantum structures, recall their basic properties, and describe the standard, and some nonstandard examples. In section 3, we describe a rich source of nonstandard examples of classical structures in the category \mathbf{Rel} of relations: every Abelian group gives a nonstandard classical structure. In fact, these are exactly the indecomposable classical structures. In section 4, we show that every classical structure in \mathbf{Rel} must arise as a direct product of indecomposables. This provides a complete characterisation of classical structures in \mathbf{Rel} . In section 5, we summarize the meaning of this characterization, and discuss the questions that it raises.

Notation. To describe relations on finite sets, we often find it convenient to use von Neumann's representation of ordinals, where $0 = \emptyset$ is the empty set, and $n = \{0, 1, \dots, n-1\}$. Moreover, the pairs $\langle i, j \rangle \in n \times n$ are often abbreviated to $ij \in n \times n$.

2 Algebras for abstraction and duality

We begin by introducing classical structures as the classical interface of quantum resources. To justify their algebraic content, we delve into a conceptual reconstruction of their role. A reader only interested in their structure should skip the next subsection.

2.1 Program abstraction and quantum computation

Abstraction is the essence of programming. The first example of program abstraction are probably Gödel's numberings of primitive recursive functions [14]. Gödel's construction demonstrated that recursive programs, specifying entire families of computations (of the values of a function for all its inputs), can be stored as data. Von Neumann later explicated this as the fundamental principle of computer architecture. Kleene, on the other side, refined the idea of program

¹ Formally, we work with relations within a given universe of sets. Each of the relational formalisms proposed in the above references will suffice for this.

abstraction into the fundamental lemma of recursion theory: the s-m-n theorem [16]. Church, finally² proposed the formal operations of variable abstraction and data application as the driving force of all computation [6]. This proposal became the foundation of functional programming.

But what is variable abstraction? What is a variable? We use them so often that it is sometimes hard to tell. A variable is adjoined to a ring, as an indeterminate element, to generate the ring of polynomials. A programmer denotes by a variable a piece of data that will only be determined when the program is run. The variable captures all the possible values of this piece of data that may arise at run time. The operation of abstraction of a variable binds all of its occurrences (within the declared scope) to the interface where its values will be read, when given. The operation of application substitutes these values for the variable.

A variable is thus a tool for propagating as-yet-unknown data through a program (or through an algebraic structure, etc). The crucial capability of such a tool is that it allows the data to be *copied* wherever they are needed, or *deleted* where they are not needed. While the classical computation, as the above early references show, was built upon this capability as its very foundation — it is a fundamental property of quantum data that they generally cannot be copied or deleted [29,11,20,1].

A classical structure formalizes this distinction: its first feature is a comonoid $X \otimes X \xleftarrow{\Delta} X \xrightarrow{\top} I$, where Δ can be used to copy and \top to delete a piece of data. A datum $I \xrightarrow{\psi} X$ is classical if it can be copied, in the sense that $\Delta\psi = \psi \otimes \psi$, and deleted, in the sense that $\top\psi = \text{id}_I$. This turns out to be exactly what is needed to support variable abstraction in monoidal categories. In the framework of dagger-monoidal categories, the requirement that abstraction preserves the dagger induces the rest of classical structure [24,21].

2.2 Frobenius algebras

Framework. Let $(\mathcal{C}, \otimes, I)$ be a monoidal category. With no loss of generality, we assume that the tensor is strictly associative and unitary, i.e. that the objects of \mathcal{C} form an ordinary monoid with respect to \otimes and I . Every monoidal category is equivalent to one which is strict in this sense. We call the arrows from I *vectors*, and write $\mathcal{C}(X) = \mathcal{C}(I, X)$.

Definition 2.1 *The structure of a Frobenius algebra X in \mathcal{C} consists of*

- an internal monoid $X \otimes X \xrightarrow{\nabla} X \xleftarrow{\perp} I$, and
- an internal comonoid $X \otimes X \xleftarrow{\Delta} X \xrightarrow{\top} I$,

² Although Church's paper appeared three years earlier than Kleene's, Church's proposal is the final step in the conceptual development of function abstraction as the foundation of computation.

such that the following diagram commutes

$$\begin{array}{ccc}
 X \otimes X & \xrightarrow{X \otimes \Delta} & X \otimes X \otimes X \\
 \Delta \otimes X \downarrow & \searrow \nabla & \downarrow \nabla \otimes X \\
 & X & \\
 X \otimes X \otimes X & \xrightarrow{X \otimes \nabla} & X \otimes X
 \end{array} \quad (\text{fro})$$

A Frobenius algebra $(X, \nabla, \Delta, \perp, \top)$ is special if its monoid and comonoid structures are normalized, in the sense that the diagram

$$\begin{array}{ccc}
 & X \otimes X & \\
 \Delta \nearrow & & \searrow \nabla \\
 X & \xrightarrow{\text{id}} & X
 \end{array} \quad (\text{spe})$$

also commutes.

Proposition 2.2 In every monoidal category, being a special Frobenius algebra is a property of the monoid (X, ∇, \perp) . More precisely, if both $(X, \nabla, \Delta_1, \perp, \top_1)$ and $(X, \nabla, \Delta_2, \perp, \top_2)$ are special Frobenius algebras, then $\Delta_1 = \Delta_2$ and $\top_1 = \top_2$.

Dually, being a special Frobenius algebra can be viewed as a property of the comonoid (X, Δ, \top) .

The monoid part (X, ∇, \perp) of a special Frobenius algebra is abelian if and only if the corresponding comonoid part (X, Δ, \top) is.

Much of the power of the Frobenius algebra structure arises from the way in which it gives rise to dualities.

Duality. A duality in a monoidal category \mathcal{C} consists of two objects and two arrows, written $(\eta, \varepsilon) : X \dashv X^*$, where

- the copairing $I \xrightarrow{\eta} X^* \otimes X$ and
- the pairing $X \otimes X^* \xrightarrow{\varepsilon} I$

are required to satisfy the equations

$$(\varepsilon \otimes X)(X \otimes \eta) = X \quad (X^* \otimes \varepsilon)(\eta \otimes X^*) = X^*$$

If every object $X \in \mathcal{C}$ has a chosen dual X^* , then the duality can be extended to a functor $(-)^* : \mathcal{C}^{op} \longrightarrow \mathcal{C}$, which maps $A \xrightarrow{f} B$ to

$$f^* : B^* \xrightarrow{\eta B^*} A^* A B^* \xrightarrow{A f B^*} A^* B B^* \xrightarrow{A^* \varepsilon} A^*$$

Frobenius algebras and dualities. Every Frobenius algebra X induces

- a pairing $\varepsilon : X \otimes X \xrightarrow{\nabla} X \xrightarrow{\top} I$, and
- a copairing $\eta : I \xrightarrow{\perp} X \xrightarrow{\Delta} X \otimes X$,

which together make X into a self-dual object, with $X^* = X$. Categorically, this means that X is adjoint to itself at the same time on the left and on the right, if the monoidal category \mathcal{C} is viewed as a bicategory with a single 0-cell. E.g., if this 0-cell is a category \mathbb{D} and $\mathcal{C} = \mathbb{D}^{\mathbb{D}}$ is the category of endofunctors $F, G \dots : \mathbb{D} \longrightarrow \mathbb{D}$, with the natural transformations as the arrows between them, and with the composition playing the role of the tensor $F \otimes G = F \circ G$, then

- the monoid $F \otimes F \xrightarrow{\nabla} F \xleftarrow{\perp} Id$ makes the functor F into a monad,
- the comonoid $F \otimes F \xleftarrow{\Delta} F \xrightarrow{\top} Id$ makes it into a comonad, and
- condition (fro) makes the pairing $\varepsilon : FF \longrightarrow Id$ and copairing $\eta : Id \longrightarrow F$ into an adjunction $F \dashv F$.

This functorial setting was first described by Lawvere [18], who characterized it by requiring the commutativity of the following diagrams of natural transformations

$$\begin{array}{ccccc}
 FFF & \xleftarrow{F\Delta} & FF & \xrightarrow{\Delta F} & FFF \\
 \nabla F \downarrow & & \downarrow \nabla & & \downarrow F\nabla \\
 FF & \xrightarrow{\top F} & F & \xleftarrow{F\top} & FF
 \end{array}
 \qquad
 \begin{array}{ccccc}
 FF & \xleftarrow{\perp F} & F & \xrightarrow{F\perp} & FF \\
 \Delta F \downarrow & & \downarrow \Delta & & \downarrow F\Delta \\
 FFF & \xrightarrow{F\nabla} & FF & \xleftarrow{\nabla F} & FFF
 \end{array}$$

and attached the name of Frobenius to such structures. The equivalent but simpler condition (fro) first appeared in Carboni and Walters' work [5], characterizing relations as a cartesian bicategory. The geometric meaning of (fro) in the category of cobordisms brought the same condition to prominence in the categorical version of Topological Quantum Field Theory [17]. Finally, its role in supporting a generic form of abstraction, on which the interface between the classical and the quantum computation turns out to be based [21], made it into an important piece in categorical Quantum Mechanics [2,10].

2.3 Classical and quantum structures

Framework. Categorical quantum mechanics actually requires a slight extension of monoidal categories: besides the monoidal structure, the category \mathcal{C} should come equipped with a functor $(-)^{\dagger} : \mathcal{C}^{op} \longrightarrow \mathcal{C}$, which is identity on the objects, and involutive on the arrows, i.e. satisfying $f^{\dagger\dagger} = f$. The arrows u such that $u^{\dagger}u = \text{id}$ and $uu^{\dagger} = \text{id}$ are called unitary. All monoidal coherences in a dagger monoidal category are required to be unitary. In the strict case, this boils down to the requirement that the symmetries are unitary, since the other coherences are already identities.

The abstract structure of a *symmetric dagger-monoidal* category $(\mathcal{C}, \otimes, I, \dagger)$ turns out to support the main constructions of quantum mechanics, normally presented using Hilbert spaces [2,26,10].

Definition 2.3 A special Frobenius algebra $(X, \nabla, \Delta, \perp, \top)$ in a symmetric dagger-monoidal category \mathcal{C} is called a *classical structure* if its monoid and its comonoid parts are

- *adjoint*, i.e. $\nabla = \Delta^\dagger$ and $\perp = \top^\dagger$
- *abelian*, i.e. $\nabla \circ (a \otimes b) = \nabla \circ (b \otimes a)$.

Interpretation. In categorical quantum mechanics, classical structures can be used to distinguish the classical resources from the quantum resources. On one hand, each classical structure extracts the *classical elements*. On the other hand, it supports the entanglement phenomena, implemented through *quantum structures*. We now recall these concepts from [10].

Definition 2.4 A quantum structure in a dagger-monoidal category is a pair (X, η) , such that $\eta : I \longrightarrow X \otimes X$ and $\eta^\dagger : X \otimes X \longrightarrow I$ make X self-dual, i.e. $(\eta, \eta^\dagger) : X \dashv X$.

Proposition 2.5 Every classical structure induces a quantum structure, with the pairing $\varepsilon : X \otimes X \xrightarrow{\nabla} X \xrightarrow{\top} I$, and the copairing $\eta : I \xrightarrow{\perp} X \xrightarrow{\Delta} X \otimes X$.

Several classical structures may induce the same quantum structure. Some quantum structures do not arise from a classical structure.

Definition 2.1. Classical elements³ for a classical structure X in \mathcal{C} are the arrows $\varphi \in \mathcal{C}(X)$ such that $\Delta\varphi = \varphi \otimes \varphi$ and $\top\varphi = \text{id}_I$.

Classical elements are thus just those vectors that can be copied and deleted. On the other hand, the entanglement capability of quantum structures is obtained by applying the copying facility of a classical structure to non-classical elements, such as the monoid unit of the classical structure itself.

2.4 Examples

The trivial example of a classical structure, present in every monoidal category, is the tensor unit I : the canonical isomorphisms $I \otimes I \cong I$ make it into a special Frobenius algebra. In the categories $(\mathbf{FVec}, \otimes, K)$ of finitely dimensional vector spaces and $(\mathbf{FHilb}, \otimes, K)$ over any field K , a choice of base $|0\rangle, |1\rangle, \dots, |n\rangle \in X$ makes each space X into a classical structure, defined by the linear operators

$$\Delta|i\rangle = |ii\rangle \qquad \top|i\rangle = 1$$

³ In the Hopf algebra theory, the elements that satisfy the same conditions are called *set-like*.

In the monoidal category $(\mathbf{Rel}, \times, 1)$, every object comes with a similar classical structure

$$\Delta(i) = \{ii\} \qquad \tau(i) = 0$$

where ij abbreviates the pair $\langle i, j \rangle \in X \times X$, and 0 is the unique element of 1. Both of these families of classical structures are induced by the cartesian structure of the category \mathbf{FSet} of finite sets, canonically embedded in \mathbf{FVec} and \mathbf{FRel} . We call them *standard* classical structures. They are characterized and analyzed in detail in [5]. In [8], it has been shown that all classical structures in \mathbf{FHilb} are standard. Very recently [9], though, Bill Edwards and Bob Coecke noticed a nonstandard classical structure $2 \times 2 \xleftarrow{\oplus} 2 \xrightarrow{\dashv} 1$ over the set $2 = \{0, 1\}$, defined as follows

$$\begin{aligned} \oplus(0) &= \{00, 11\} & \dashv(0) &= \{0\} \\ \oplus(1) &= \{01, 10\} & \dashv(1) &= 0 \end{aligned}$$

Note that both the standard classical structure $2 \times 2 \xleftarrow{\Delta} 2 \xrightarrow{\top} 1$ and the nonstandard classical structure $2 \times 2 \xleftarrow{\oplus} 2 \xrightarrow{\dashv} 1$ induce the same quantum structure $1 \xrightarrow{\eta} 2 \times 2$, relating 0 with 00 and 11. But this turns out to be an exception. E.g., a little trial and error leads to the following nonstandard classical structure $3 \times 3 \xleftarrow{\oplus} 3 \xrightarrow{\dashv} 1$, where $3 = \{0, 1, 2\}$

$$\begin{aligned} \oplus(0) &= \{00, 12, 21\} & \dashv(0) &= \{0\} \\ \oplus(1) &= \{22, 01, 10\} & \dashv(1) &= 0 \\ \oplus(2) &= \{11, 02, 20\} & \dashv(2) &= 0 \end{aligned}$$

The induced quantum structure $1 \xrightarrow{\eta} 3 \times 3$ now relates 0 with 00, 12 and 21, whereas the standard one relates 0 with 00, 11 and 22. Soon we shall see how this comes about.

2.5 Representations of classical structures

By definition 2.1, classical structures are given as *internal* algebras. They are thus defined in any dagger monoidal category \mathcal{C} . However, some parts of the analysis of classical structures is simpler with a more concrete representation.

According to proposition 2.2, a classical structure $(X, \nabla, \Delta, \perp, \tau)$ is completely determined by the monoid part (X, ∇, \perp) . This internal monoid can be represented as a monoid of endomorphisms on X in \mathcal{C} , as follows: proceeding as follows:

- first *externalize* the internal monoid (X, ∇, \perp) in \mathcal{C} as an ordinary monoid of vectors $(\mathcal{C}(X), \cdot, \perp)$, by setting

$$\varphi \cdot \psi = \nabla \circ (\varphi \otimes \psi)$$

– then represent every vector $\varphi \in \mathcal{C}(X)$ as an action $\mathcal{I}\varphi$ over X , by

$$\begin{aligned} \mathcal{I} : \mathcal{C}(X) &\longrightarrow \mathcal{C}(X, X) \\ (I \xrightarrow{\varphi} X) &\longmapsto (X \xrightarrow{\varphi \otimes X} X \otimes X \xrightarrow{\nabla} X) \end{aligned}$$

This second step can be viewed either as a generalization of Cayley’s group representation to monoids, or as a special case of Yoneda’s embedding of categories.

Proposition 2.6 *The monoid $(\mathcal{C}(X), \cdot, \perp)$ is isomorphic with the submonoid of $(\mathcal{C}(X, X), \circ, \text{id})$ which consists of the endomorphisms $f : X \rightarrow X$ such that $f \circ (a \cdot x) = (f \circ a) \cdot x$ holds for all $a, x \in \mathcal{C}(X)$.*

This allows representing any monoid as a monoid of endomorphisms. But those monoids that come from a classical structure carry more. As observed in [8], and further explored in [28], the externalisation of every Frobenius algebra, and hence every classical structure, is also a \star -algebra. The categorical presentations [15, 28] of the antilinear operation \star involve the formal duals, as spelled out in sec. 2.2.

Definition 2.7 *An internal \star -monoid in a monoidal category \mathcal{C} is a structure $(X, X^*, \nabla, \perp, \star)$ where*

- (X, ∇, \perp) is internal monoid
- X^* is a dual of X , and
- $\star : X \cong X^*$ is an isomorphism (always unitary).

We write $\varphi^* = \star \circ \varphi \in \mathcal{C}(X^*)$ for $\varphi \in \mathcal{C}(X)$.

A \star -monoid homomorphism $f : X \longrightarrow Y$ is a monoid homomorphism which moreover preserves the \star , in the sense that

$$\begin{array}{ccc} X^* & \xrightarrow{f_*} & Y^* \\ \star \downarrow & & \downarrow \star \\ X & \xrightarrow{f} & Y \end{array}$$

commutes.

Proposition 2.8 *The monoid $(\mathcal{C}(X), \cdot, \perp)$ induced by a classical structure (X, ∇, \perp) comes with an involution*

$$\begin{aligned} (-)^\star : \mathcal{C}(X) &\longrightarrow \mathcal{C}(X) \\ (I \xrightarrow{\varphi} X) &\longmapsto (I \xrightarrow{\eta} X \otimes X \xrightarrow{\varphi^\dagger \otimes X} X) \end{aligned}$$

This involution is preserved by the representation $\mathcal{I} : \mathcal{C}(X) \longrightarrow \mathcal{C}(X, X)$, in the sense that $\mathcal{I}(\varphi^\star) = (\mathcal{I}\varphi)^\dagger$ holds.

3 Simple classical structures in Rel

In the rest of the paper, we explore and characterize classical structure in a category Rel of sets and relations. Any of its formalizations (some mentioned in the Introduction) will do. Computationally, relations are usually viewed as denotations of nondeterministic programs: a binary relation $R : A \multimap B$ is the input/output relation of a program, which may output b when given an input a whenever aRb holds [19].

3.1 Meaning of (spe) in Rel

On the other hand, the isometry condition (spe) here means that the relation $\nabla : X \times X \multimap X$ is single-valued and surjective on X , i.e.

$$\begin{aligned} \forall x, y, u, v \in X. x, y \in \nabla(u, v) &\implies x = y \\ \forall x \in X \exists uv \in X. x &= \nabla(u, v) \end{aligned}$$

Equivalently, (spe) means that $\Delta = \nabla^{op} : X \multimap X \times X$ injects X into parts of $X \times X$ and is total on X .

$$\begin{aligned} \forall x, y \in X. \Delta(x) \cap \Delta(y) \neq 0 &\implies x = y \\ \forall x \in X. \Delta(x) &\neq 0 \end{aligned}$$

3.2 Meaning of (fro) in Rel

In the monoidal category $(\text{Rel}, \times, 1)$, the monoid action $\nabla : X \times X \multimap X$ is a relation, which assigns to every pair $x, y \in X$ a set $\nabla(x, y) \subseteq X$. The Frobenius condition (fro) becomes

$$\begin{aligned} \{\langle x, y \rangle \mid \nabla(i, j) \cap \nabla(x, y) \neq 0\} &= \{\langle x, \nabla(y', j) \rangle \mid i \in \nabla(x, y')\} \\ &= \{\langle \nabla(i, x'), y \rangle \mid j \in \nabla(x', y)\} \end{aligned}$$

This must be satisfied for all $i, j \in X$.

3.3 Meaning of (fro) \wedge (spe) in Rel

Notation. Since $\nabla(u, v)$, according to (spe), has at most one element, ∇ is a partial operation. It is thus convenient to write it in the infix form whenever it is defined, i.e. $u \nabla v = \nabla(u, v) \neq 0$.

The condition $\nabla(i, j) \cap \nabla(x, y) \neq 0$ now becomes $i \nabla j = x \nabla y$ and $i \in x \nabla y$ means that $i = x \nabla y$. The Frobenius condition thus boils down to

$$\begin{aligned} \{\langle x, y \rangle \mid i \nabla j = x \nabla y\} &= \{\langle x, y' \nabla j \rangle \mid i = x \nabla y'\} \\ &= \{\langle i \nabla x', y \rangle \mid j = x' \nabla y\} \end{aligned} \tag{1}$$

This characterisation provides a rich source of classical structures.

Proposition 3.1 *Every abelian group (X, ∇, \perp) in Set induces a classical structure in Rel.*

Proof. If (X, ∇, \perp) is a group, then (1) is satisfied by $x' = j \nabla y^{-1}$ and $y' = x^{-1} \nabla i$. \square

Remark. The nonstandard classical structures described in section 2.4 are easily seen to arise from the groups \mathbb{Z}_2 and \mathbb{Z}_3 .

3.4 Simplicity

Definition 3.2 A classical structure induced by an abelian group is called simple.

Proposition 3.3 A simple classical structure, viewed as a comonoid, has only trivial subobjects. More precisely, any simple classical structure Y is the range of exactly two comonoid monomorphisms:

- the empty relation $O : \emptyset \multimap Y$ and
- the chaotic relation $I : 1 \multimap Y$, where $0 \in 1$ is related to every $y \in Y$.

Proof. The fact that the empty relation $O : \emptyset \multimap Y$ is a comonoid monomorphism is easily checked. We prove that the only nonempty subobject of a simple classical structure Y is the chaotic relation $I : 1 \multimap Y$.

A relation $R : X \multimap Y$ is a monomorphism in \mathbf{Rel} if and only if the induced map

$$\begin{aligned} \wp R : \wp X &\longrightarrow \wp Y \\ U &\longmapsto \{y \in Y \mid \exists x \in U. xRy\} \end{aligned}$$

is injective. This implies

$$\forall x \in X \exists y \in Y. xRy \tag{2}$$

or else $\wp R\{x\} = \wp R\emptyset$.

On the other hand, given the comonoids X and Y in \mathbf{Rel} , unfolding the statement that a relation $R : X \multimap Y$ is a comonoid homomorphism says that

$$x\Delta\langle s, t \rangle \wedge y\Delta\langle u, v \rangle \implies (xRy \iff sRu \wedge tRv) \tag{3}$$

The claim is now that (2) and (3) imply

$$\forall x \in X \forall u \in Y. xRu \tag{4}$$

This claim is proven by instantiating (3) to $s = x$, $t = \perp$, with an arbitrary u , and $v = y - v$. The left-hand side of (3) is then satisfied, since $x\Delta\langle x, \perp \rangle$ holds by definition, and $y\Delta\langle u, v \rangle$ means $y = u + v$, for any simple Y . Together with (2), this instance of (3) implies xRu , i.e. (4).

Thus $\wp R\{x\} = Y$ holds for all $x \in X$. Since $\wp R$ is injective, this implies that X has at most one element, to which all of Y is related. The relation R is thus chaotic, as claimed. \square

4 Classification of classical structures in Rel

4.1 \star -algebras in Rel

In section 2.5, we saw that every classical structure induces a \star -algebra. In Rel, this restricts them to a very small family. The decomposition of Frobenius algebras into simple subalgebras follows.

Proposition 4.1 *The representation $\Upsilon : X \longrightarrow \text{Rel}(X, X)$ maps the elements of any classical structure X in Rel to partial bijections.*

Proof. We saw in section 3 that classical structures in Rel are partial monoids, in the sense that $x \nabla y$ has at most one element. This means that for every $y \in X$ $\Upsilon y : X \multimap X$ is a partial map.

Since $\Upsilon : X \longrightarrow \text{Rel}(X, X)$ is a \star -representation, $\Upsilon(y^\dagger) = (\Upsilon y)^{op}$ is also a partial map. But a relation $R \in \text{Rel}(X, X)$ such that both R and R^{op} are partial maps

$$\begin{aligned} xRy \wedge xRy' &\implies y = y' \\ xRy \wedge x'Ry &\implies x = x' \end{aligned}$$

must be a partial bijection. In words, for every x there is at most one y such that xRy ; and for every y there is at most one x such that xRy . \square

4.2 The main results

Proposition 4.2 *Every special Frobenius algebra in Rel is a biproduct of special Frobenius algebras where the unit is a singleton.*

Terminology. The biproduct of sets A and B in Rel is simply their disjoint union $A + B$. This means that it is at the same time their product, and their coproduct.

Proof of proposition 4.2. Let the unit $\perp \in \text{Rel}(X)$ of the special Frobenius algebra X be $\perp = \{\phi_j\}_{j \in J}$. We claim that there is a partition

$$X = \bigcup_{j \in J} X_j$$

such that

$$X_k \cap X_\ell \neq 0 \implies k = \ell$$

and such that for every $j \in J$ the partial bijection $\Upsilon\phi_j : X \multimap X$ is just the identity on X_j .

To prove this, note that

- $\Upsilon\perp = \text{id}_X$,
- $\Upsilon\phi_j \subseteq \Upsilon\perp$

- if $a \in X_j \cap X_k$, then $a\mathcal{T}\phi_j = a = a\mathcal{T}\phi_k \implies j = k$, because $a\mathcal{T} : X \dashrightarrow X$ must also be a partial bijection, as demonstrated in proposition 4.1.

Thus the domains of $\mathcal{T}\phi_j$ must cover X , and they must be disjoint.

Now we claim that (X_j, ∇_j, \perp_j) is a submonoid of (X, ∇, \perp) . This means that for every $x \in X_j$, the partial bijection $\mathcal{T}x : X \dashrightarrow X$ restricts to a bijection $\mathcal{T}_jx : X_j \longrightarrow X_j$.

Suppose that for $x, y \in X_j$ it happens that $x\nabla y \in X_k$. That would mean that $y\nabla\phi_k$ must be defined, because $x\nabla y = (x\nabla y)\nabla\phi_k = x\nabla(y\nabla\phi_k)$. But then $y = y\nabla\phi_k \in X_k$, and we get $y \in X_j \cap X_k$. We have seen above that this implies $j = k$. \square

Proposition 4.3 *Suppose that $(X, \Delta, \nabla, \top, \perp)$ is a classical structure in **Rel**, such that the unit $\perp : 1 \longrightarrow X$ is a function, i.e. a single element of X . Then the monoid (X, ∇, \perp) must be an abelian group in **Set**.*

Proof. We first show that the monoid part of every classical structure X in **Rel** must admit the inverses, as soon as it satisfies the assumptions of the proposition. More precisely, the claim is that condition (1) from section 3.3, together with the assumption that the unit is a singleton $\perp \in X$, implies that for every $k \in X$ there is $k^{-1} \in X$ such that $k\nabla k^{-1} = k^{-1}\nabla k = \perp$.

First consider condition (1) for $i = k$ and $j = \perp$. For the pair $\langle x, y \rangle = \langle \perp, k \rangle$, the second equation gives $x' \in X$ such that $\perp = x'\nabla k$. Dually, (1) also holds for $i = \perp$ and $j = k$. For the pair $\langle x, y \rangle = \langle k, \perp \rangle$, the first equation gives $y' \in X$ such that $\perp = k\nabla y'$. Since $x' = x'\nabla \perp = x'\nabla(k\nabla y') = (x'\nabla k)\nabla y' = \perp\nabla y' = y'$, we can set $k^{-1} = x' = y'$.

To see that (X, ∇, \perp) is a group, it remains to be shown that the operation ∇ is total, i.e. that $k\nabla \ell$ is defined for all $k, \ell \in X$. To see that this is the case, note that in each of the equations $\ell = (k^{-1}\nabla k)\nabla \ell = k^{-1}\nabla(k\nabla \ell)$, the left-hand side is defined if and only if the right-hand side is defined. Hence $k\nabla \ell$ must be defined.

Since the monoid operation $\nabla : X \times X \longrightarrow X$ is total, and every element $k \in X$ has an inverse k^{-1} , we conclude that (X, ∇, \perp) is indeed a group. \square

Given an arbitrary classical structure $(X, \Delta, \nabla, \top, \perp)$ in **Rel**, we can now first apply proposition 4.2 to decompose it as a biproduct

$$X = \sum_{j \in J} X_j$$

of classical structures $(X_j, \Delta_j, \nabla_j, \top_j, \perp_j)$ where each \perp_j is a singleton. By proposition 4.3, each of these classical structures is a group. Hence the final result:

Theorem 4.4 *Every special Frobenius algebra in **Rel** is a biproduct (disjoint union) of groups. Every classical structure in **Rel** is a biproduct of abelian groups.*

Using this result, we can now effectively enumerate all classical structures in **Rel** with a given number of elements.

4.3 Examples of classical structures

Any classical structure over a set of n elements can thus be constructed by choosing

- a partition $n = \sum_j n_j$, where $j \geq 1$,
- an abelian group X_j of order n_j for each n_j .

For $n = 2$, there are just two partitions: $n = 1 + 1$ and $n = 2$. Since there is just one group with a single element, and just one group with 2 elements, these two partitions each determine a unique classical structure. They were described in section 2.4.

For $n = 3$, besides $n = 1 + 1 + 1$ and $n = 3$, we can also write $n = 1 + 2$. The first two partitions give the classical structures described in section 2.4. The nonstandard one comes from \mathbb{Z}_3 . The third classical structure is the disjoint union $\mathbb{Z}_1 + \mathbb{Z}_2$.

For $n = 4$ there are five partitions. It is easy to see the pattern: e.g., $n = 2 + 2$ induces the classical structure $\mathbb{Z}_2 + \mathbb{Z}_2$, whereas $n = 1 + 3$ induces $\mathbb{Z}_1 + \mathbb{Z}_3$. Since there are two groups with 4 elements, \mathbb{Z}_4 and the Kleinian group $D_4 = \mathbb{Z}_2 \times \mathbb{Z}_2$, the trivial partition $n = 4$ induces two different classical structures. They both have the same classical element, consisting of all of $n = 4$; but they induce different quantum structures, entangling each element with its group inverse. Since each element of D_4 is its own inverse, its quantum structure coincides with the one induced by the standard classical structure $\mathbb{Z}_1 + \mathbb{Z}_1 + \mathbb{Z}_1 + \mathbb{Z}_1$. In any case, there are exactly 6 different classical structures with 4 elements.

For $n = 5$, there are 7 different partitions, and they induce 8 different classical structures. E.g., the partition $n = 2 + 3$ corresponds to the classical structure $\mathbb{Z}_2 + \mathbb{Z}_3$, with the classical elements $\{0, 1\}$ and $\{2, 3, 4\}$. The quantum structure is $\eta = \{00, 11, 22, 34, 43\}$.

For $n = 6$, there are 11 partitions. There are 2 groups with 6 elements, but only the cyclic one is abelian. . .

In all cases, the classical elements of a classical structure are just the underlying sets of its constituent groups. They do not depend on the actual structure of the groups. This structure is, however, reflected in the induced quantum structure, which entangles each element with its group inverse.

5 Conclusions and future work

We classified classical structures in \mathbf{Rel} , and found that many are nonstandard. They also induce many nonstandard quantum structures $I \xrightarrow{\eta} X \times X$ in \mathbf{Rel} . If X is a group, then η entangles each $a \in X$ with its inverse a^{-1} . Moreover, each of the nonstandard classical structures induces a nonstandard abstraction operator κx , binding the variable x in the polynomial relations $\varphi(x) \in \mathbf{Rel}[x]$. For monoidal categories in general, such operations and their meaning were analyzed in [24]. In \mathbf{Rel} in particular, the situation seems rather curious. While the nonstandard classical structures support specifying relational polynomials, as

nondeterministic programs with nonstandard variables — proposition 3.3 says that there are only trivial classical elements to be substituted for these variables. The distinctions of the elements belonging to the same group within a nonstandard classical structure turn out to be classically indistinguishable. However, this indistinguishability, observed through a different classical structure, can be used as a computational resource. Indeed, switching between the different classical structures in order to use this resource is the essence of some of the most important quantum algorithms. The interesting structural repercussions of this method within the relational view of nondeterministic computation need to be further explored in future work.

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